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LETTER TO THE EDITOR

High-temperature spontaneous magnetization of models with pure triplet interactions

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Abstract. A configurational proof is given that at high enough temperatures the triplet model on the triangular lattice has zero spontaneous magnetization. On the face-centred cubic lattice the spontaneous magnetization is shown to be nonzero for all finite temperatures. This seems to imply the absence of a phase transition of the conventional type in zero field.

A great deal of interest has recently been shown in models of magnetic systems with multispin interactions. In this letter we consider one of these models, the triplet model, in which the interaction energy is the product of the spin variables at the vertices of triangles of the lattice. We concentrate on the triangular and face-centred cubic (FCC) lattices, referring to work on other lattices when appropriate.

For each of these models we investigate the occurrence at high temperatures of a zero-field magnetization. Merlini *et al* (1973) have shown that the triangular lattice triplet model has a nonzero spontaneous magnetization at sufficiently low temperatures and have given a lower bound for the critical temperature. We have found it worthwhile to use configurational arguments to investigate the high-temperature region.

Our results contradict one of the basic assumptions made by Wood and Griffiths (1974a,b) and Griffiths and Wood (1973, 1974) in analysing their low-temperature series. They implicitly assume that the magnetization goes to zero as $(T_c - T)^{\beta}$. This assumption is invalid on the FCC lattice with triplet interactions. It remains invalid if non-negative two-spin and four-spin interactions are added. The assumption is also invalid on the triangular lattice if triplet interactions are combined with positive two-spin interactions. For the body-centred cubic (BCC) lattice with first and second nearest-neighbour bonds the same results as for the FCC lattice are true. For each of the above models similar comments apply to the assumption that the magnetization goes to zero as $H^{1/\delta}$ along the critical isotherm.

We consider pure triplet interactions of strength J_3 on the triangular lattice and expand the logarithm of the partition function, ln Z, about $T = \infty$ as a power series in the high-temperature variables v ($=\tanh J_3/kT$) and τ ($=\tanh mH/kT$). The coefficient of $v^n\tau^r$ corresponds to the number of weak embeddings (Sykes *et al* 1966) in the lattice of a configuration containing n triangles with r odd vertices: an odd vertex is here defined as a vertex that is common to an odd number of triangles. The high-temperature spontaneous magnetization is given by

$$m^{-1}I(v) = \frac{\partial}{\partial \tau} \ln Z(v, \tau)|_{\tau=0}.$$

Hence to show that the magnetization is zero we need to show that no configurations with exactly one odd vertex are embeddable in the lattice. We divide the lattice into three sublattices A, B, C as shown in figure 1(*a*). Note that any elementary triangle has one vertex on each of the sublattices. We now show that a graph containing an odd (even) number of triangles has an odd (even) number of odd vertices on each sublattice.

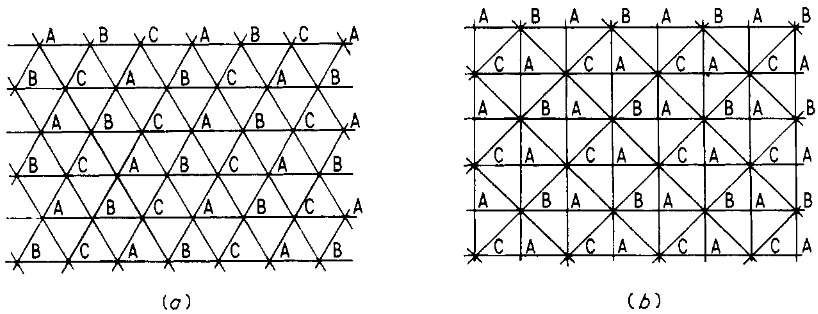

Figure 1. Three-sublattice division for: (*a*) the triangular lattice; (*b*) the Union Jack lattice.

This is obviously true for $n = 1$ and addition of a triangle to any graph changes the number of odd vertices on each sublattice by ± 1; hence it is true for all n by induction. Since three positive integers, all odd or all even, cannot sum to unity, we conclude that no graph may contain exactly one odd vertex. This result of a vanishing high-temperature magnetization is indicated by the work of Merlini *et al* (1973) and by the complete expression for a spontaneous magnetization vanishing at T_c, proposed by Baxter *et al* (1975).

The above proof applies, using the sublattice division of figure 1(*b*), to the Union Jack lattice triplet model whose free energy was obtained by Hinterman and Merlini (1972).

We cannot however obtain an appropriate three-sublattice division of the FCC lattice and in fact there are configurations with only one odd vertex (figure 2). This indicates that the magnetization does not vanish and suggests the absence of a phase transition.

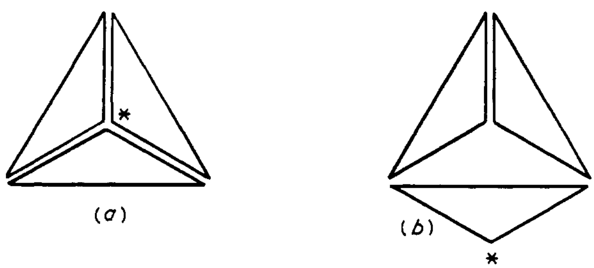

Figure 2. The graphs containing three triangles and one odd vertex which are embeddable in the FCC lattice. The odd vertex is indicated by the asterisk.

The connection between the occurrence of a phase transition and the possibility of a three-sublattice division is also suggested by symmetry arguments. If the three-sublattice division is possible then the zero-field free energy is unchanged by reversing the signs of all spins on two of the sublattices. This can be done in three ways so that the $H=0$ line is a line of quadruple points terminating at a critical point. On the FCC lattice however there is a unique ground state. It appears that the possibility of a phase transition associated with a spontaneous symmetry breaking is precluded by the lack of symmetry.

In fact for the FCC lattice the presence of configurations such as those in figure 2 means that we can place a rigorous, nonzero, lower bound on the spontaneous magnetization. We consider a single tetrahedron, simplex S_4, as a finite cluster. This contains the graph of figure 2(a) and we calculate the zero-field spin expectation value of the cluster as

$$\mathrm{Tr}(\sigma_i \exp(-\beta\mathscr{H}))/\mathrm{Tr}(\exp(-\beta\mathscr{H})) = v^3.$$

Since all the interactions are positive we can use Griffiths' (1967) second inequality, as generalized by Kelly and Sherman (1968), which states that all spin expectation values are non-decreasing functions of interaction strengths.

For any particular spin its expectation value for a given set of interactions will be bounded below by the expectation value obtained for a subset of these interactions. For a homogeneous lattice we can equate any spin expectation value to the magnetization. For the FCC lattice the magnetization is bounded below by v^3, the expectation value of a spin on an isolated tetrahedron.

Improved bounds can be found by taking larger subsystems of the FCC lattice. Considering a cluster X of four tetrahedra, meeting only at vertex A, leads to

$$I_{\mathrm{FCC}}/m = \langle\sigma_{\mathrm{A}}\rangle \geqslant \langle\sigma_{\mathrm{A}}\rangle_{\mathrm{X}} = 4(v^3+v^9)/(1+6v^2+v^{12}).$$

These bounds will also apply to the BCC lattice with triplet interactions around triangles of two nearest-neighbour bonds, and a second-neighbour bond. The bounds also apply when positive two-spin and four-spin interactions are added to these triplet interactions. In this case however it is easy to obtain better bounds than those above. Even on the Union Jack and triangular lattices systems with mixed two-spin and three-spin interactions will have nonzero magnetization in zero field.

In systems with two-spin interactions only along a subset of bonds the lattice may be inhomogeneous and the magnetization may not be the appropriate order parameter. Symmetry arguments like those above can be useful in understanding the behaviour of such complicated systems and in determining an appropriate order parameter. We have not been able to find any alternative order parameter for BCC and FCC triplet models and so we believe that, not only do our magnetization bounds invalidate the magnetization analysis of Wood and Griffiths (1974a,b) and Griffiths and Wood (1974), but that our bounds also strongly suggest the absence of any phase transition of the conventional type in zero field for BCC and FCC triplet models. We cannot completely exclude the possibility of phase transitions of a more general type than that implied by the analysis of Wood and Griffiths, but it seems more probable that any transitions that do occur do so only in negative field.

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